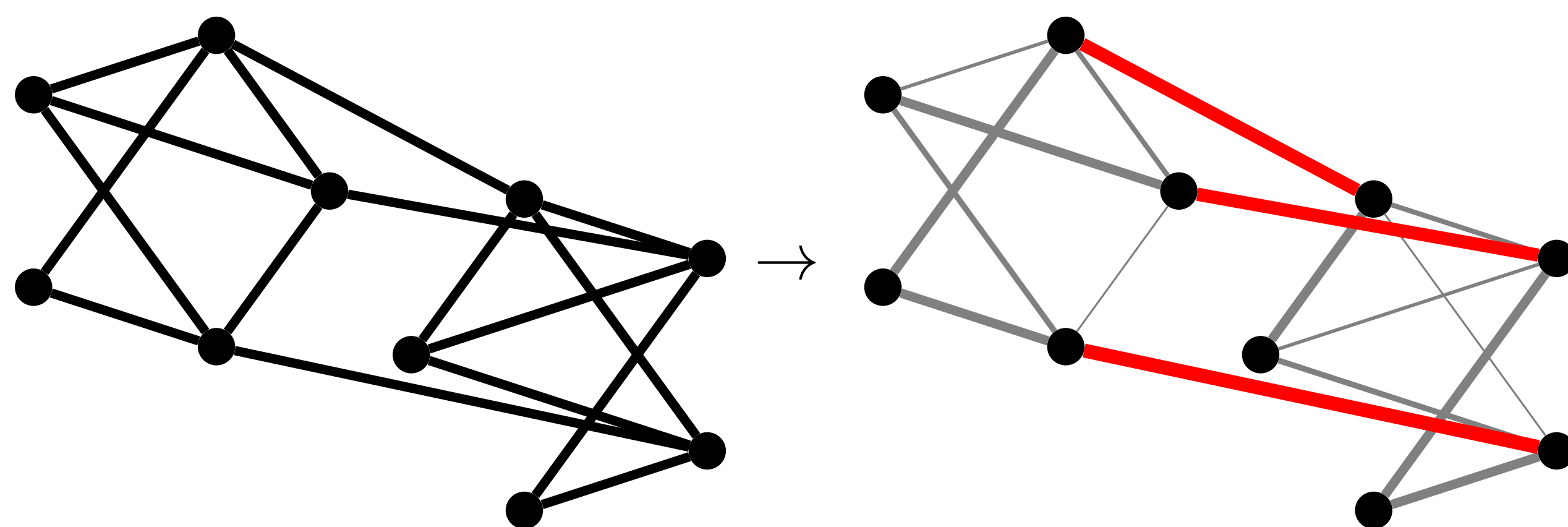


Background

- How can one rank n items given the outcomes of random pairwise comparisons between them? The comparison graph $\mathcal{G} = ([n], \mathcal{E})$ may be sparse (not every pair is compared).
- Assume that comparison outcomes are i.i.d. random variables $Z_{ij}^{(m)} \sim \text{Bernoulli}(p_{ij})$ for all $(i, j) \in \mathcal{E}$ and $m \in [k]$.
- Bradley-Terry-Luce (BTL) model:** Each item i has score $\alpha_i > 0$ such that j beats i with probability $p_{ij} \triangleq \alpha_j / (\alpha_i + \alpha_j)$.
- BTL estimation:** Estimate canonical (normalized) score vector $\pi \triangleq (\alpha_1, \alpha_2, \dots, \alpha_n) / \sum_{i=1}^n \alpha_i$ given empirical probabilities $\hat{p}_{ij} \triangleq \sum_{m=1}^k Z_{ij}^{(m)} / k$ for all $(i, j) \in \mathcal{E}$.
- Spectral method** (rank centrality): Construct row-stochastic matrix $\hat{S} \in \mathbb{R}^{n \times n}$ such that $\hat{S}_{ij} \propto \hat{p}_{ij}$ for all $(i, j) \in \mathcal{E}$ then take the stationary distribution to obtain the estimate $\hat{\pi}$.
 - This works because if we construct S analogously with the true probabilities p_{ij} , its principal eigenvector will be exactly π .
- The performance of such algorithms depends on the **spectral properties** of the graph, such as:
 - $\lambda_2(S)$: The second largest eigenvalue of S
 - $\lambda_{n-1}(L)$: The second smallest eigenvalue of the Laplacian of $\mathcal{G}$
- Optimal entrywise (i.e., ℓ^∞) error bounds are known for spectrally “nice” graphs such as Erdős-Rényi graphs $(\Theta(\sqrt{\log(n)/(npk)}))$, but the best known bound for general graphs is not tight.



Before reweighting:
poor (small) spectral gap

After reweighting:
better spectral gap

Main Results

- We derive ℓ^∞ error bounds for spectral BTL estimation on **semi-random graphs**.
- Semirandom graph: Sample each edge (i, j) independently with probability $q_{ij} \geq p$, where p is fixed and satisfies $np \geq \log(n)$.

Theorem (Entry-Wise Error Bounds for Unweighted Spectral Method)

If $1 - |\lambda_2(S)| \leq \gamma$ holds with high probability for some constant $0 < \gamma < 1$, there exist constants $c_0 > 1$ and $c_1, c_2 > 0$ such that for all p satisfying $np \geq c_0 \log(n)$, the approximate score vector $\hat{\pi}$ estimated by the spectral method satisfies

$$\frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \leq \left(\frac{c_1}{\gamma} + c_2 \right) \sqrt{\frac{\log(n)}{npk}} \quad (1)$$

for sufficiently large k and n with high probability under certain regularity conditions.

- A semirandom graph provides at least as much information as an Erdős-Rényi graph with the same p .
- Despite this, an adversary can choose q_{ij} such that $\lambda_2(S)$ is minimized to hurt the performance of the spectral method.
- Edge reweighting:** “undo” the adversary’s actions by reducing the weights of overrepresented edges.

Theorem (Entry-Wise Error Bounds for MMWU-Weighted Spectral Method)

If $np \geq c_0 \log(n)$ for some constant $c_0 > 1$, there exist some constants $c_1, c_2, c_3 > 0$ such that the approximate score vector $\hat{\pi}$ estimated by the MMWU-weighted spectral method satisfies

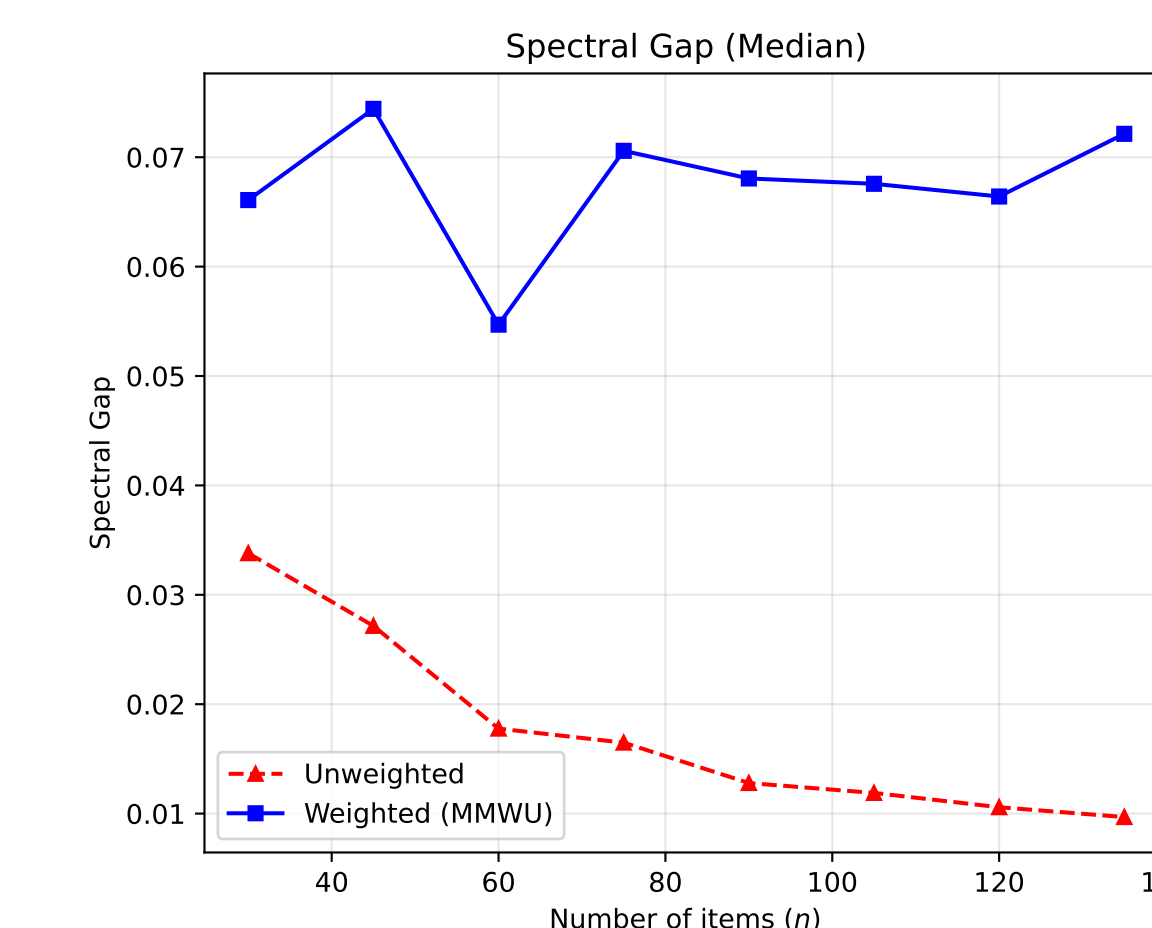
$$\frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \leq c_1 \sqrt{\frac{\log(n)}{npk}} + c_2 \sqrt{\frac{n \log(n)}{(np)^3 k}} \quad (2)$$

for sufficiently large k with high probability.

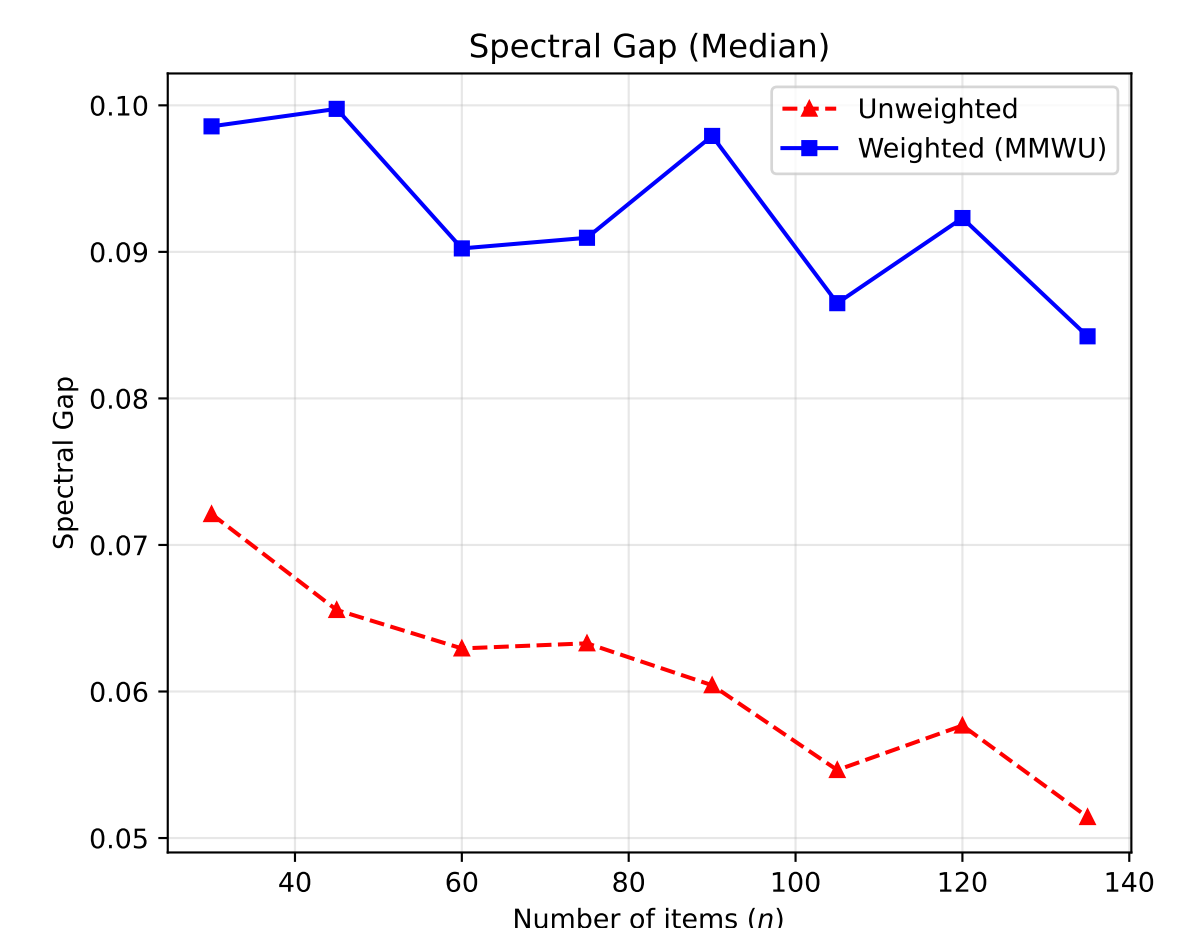
- With Matrix Multiplicative Weight Update (MMWU) weighting, the spectral method’s performance no longer depends on $\lambda_2(S)$.

Experiments

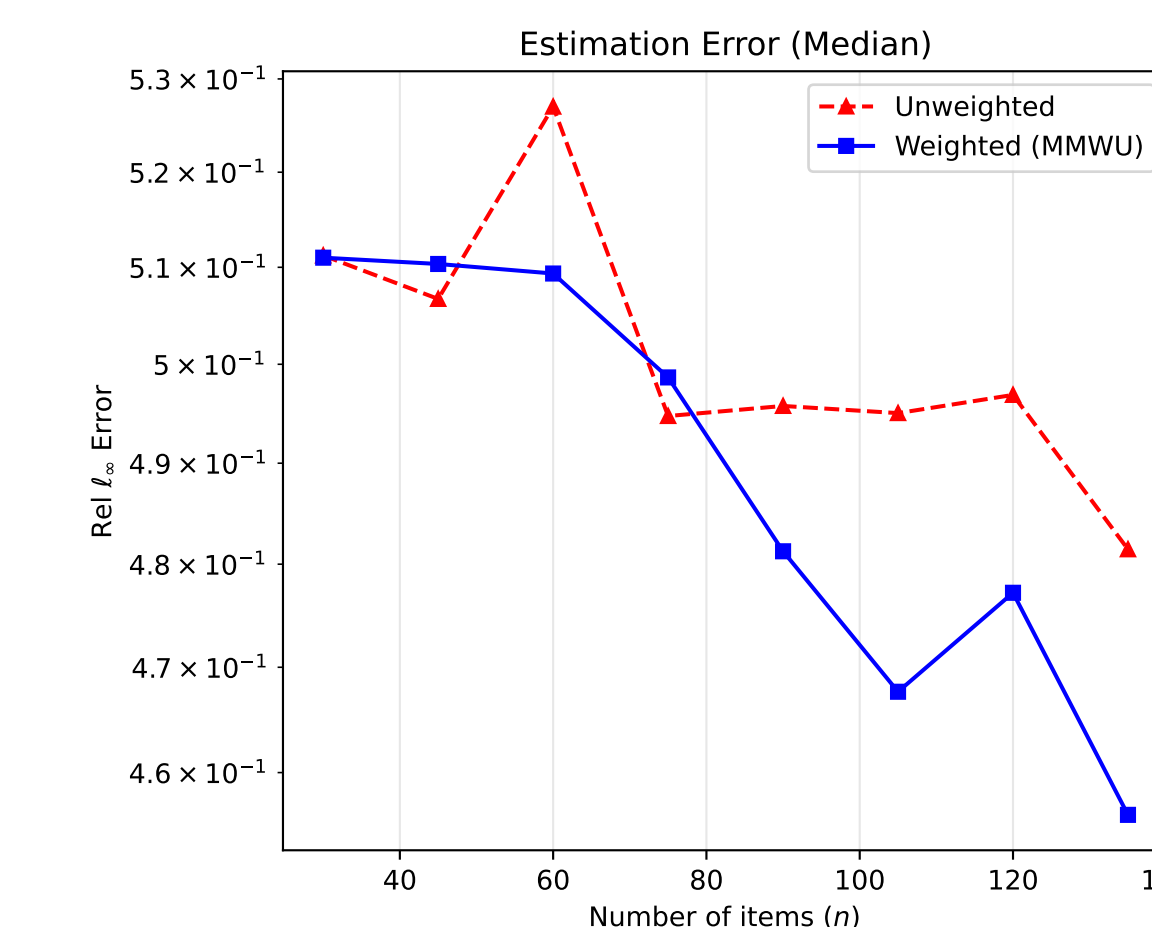
- Experiment 1 (stochastic block model):** Reweighting helps improve the performance of spectral BTL estimation.
- Experiment 2 (Erdős-Rényi model):** Reweighting does not make a big difference in this case, as the observation graph is already spectrally well-behaved.



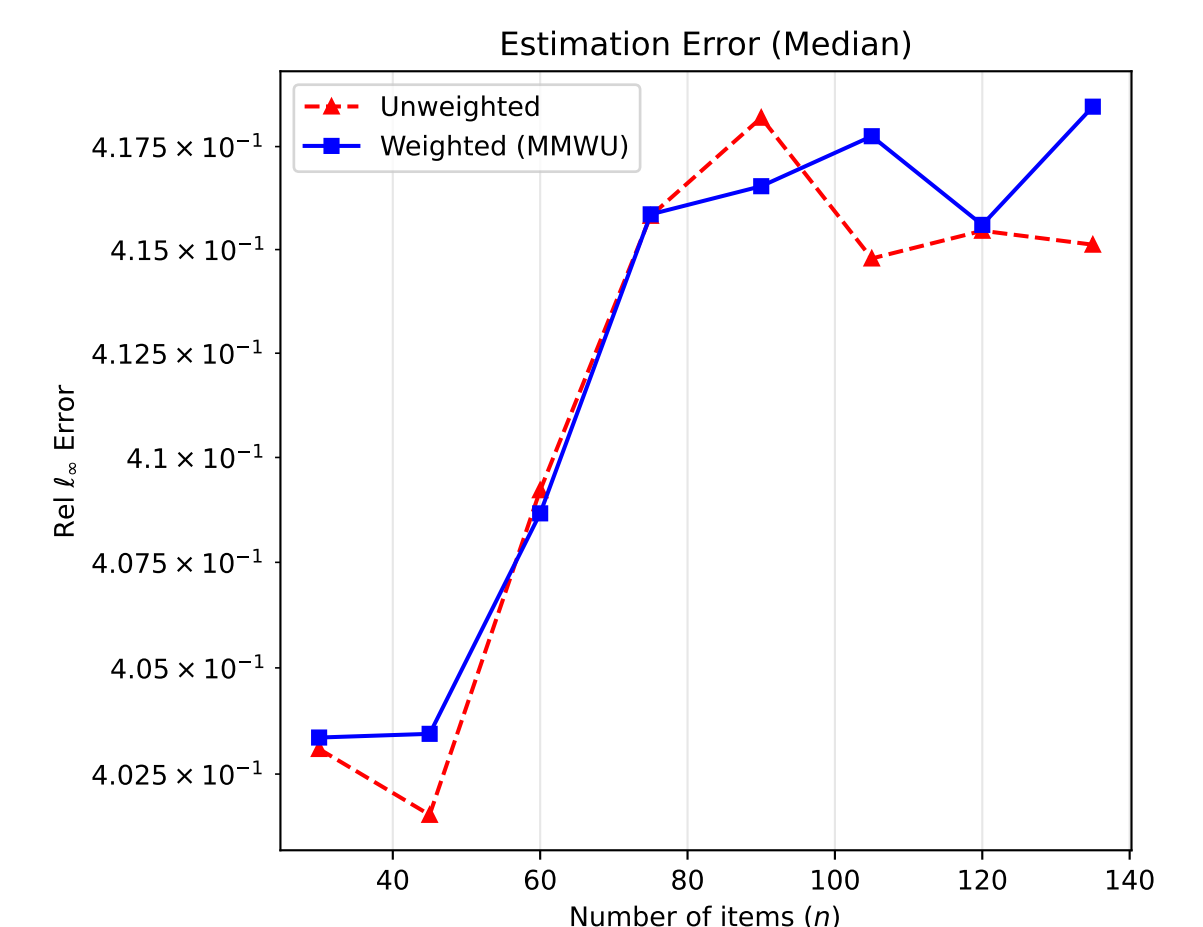
(a) Spectral gap (Experiment 1)



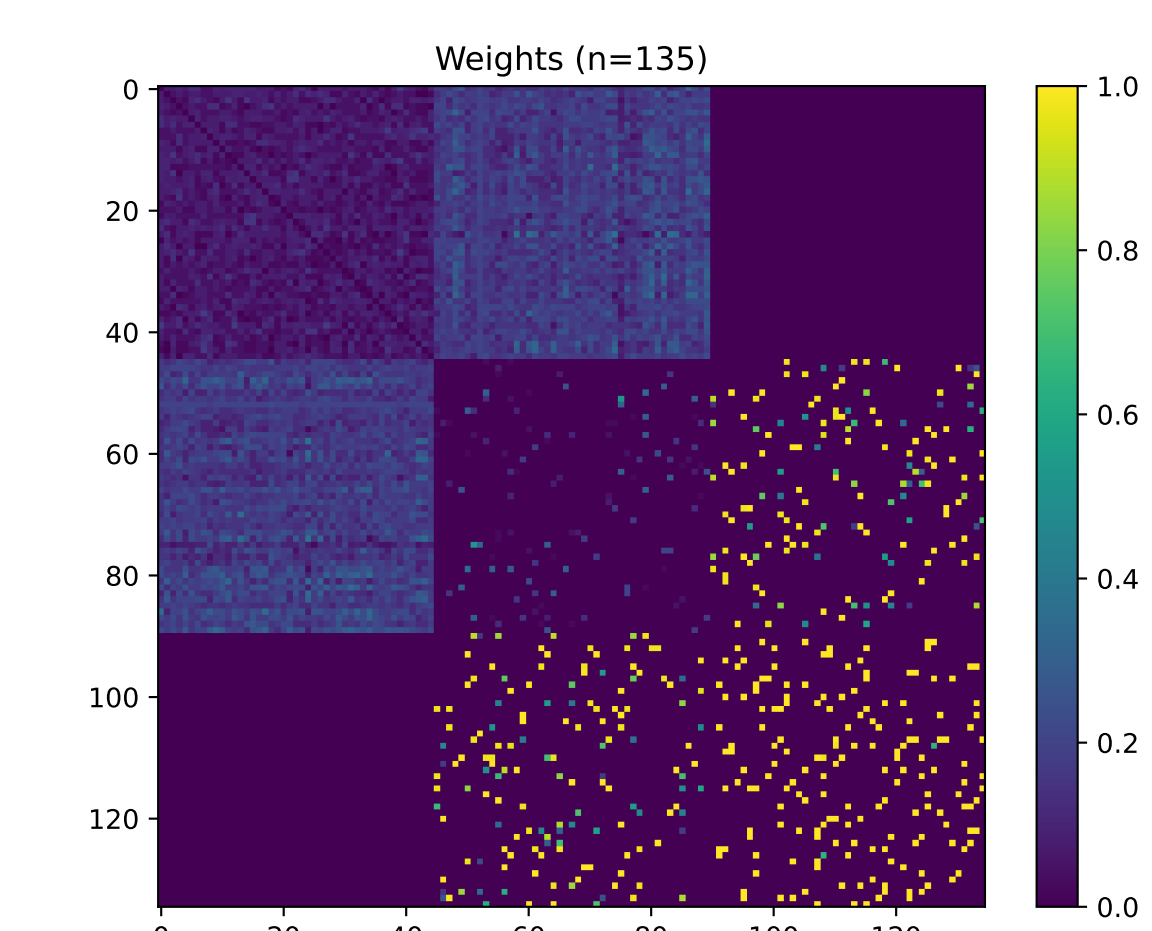
(d) Spectral gap (Experiment 2)



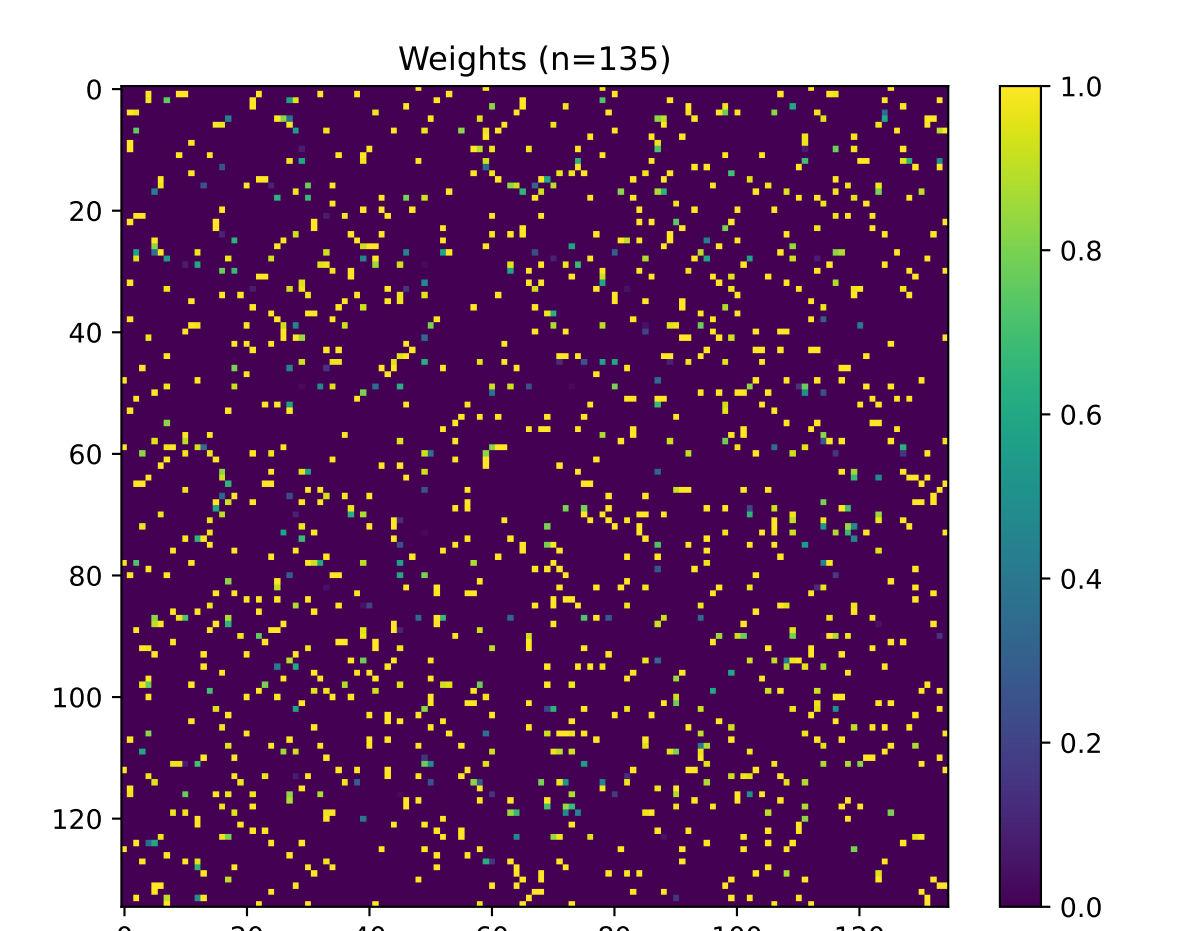
(b) Relative ℓ^∞ -error (Experiment 1)



(e) Relative ℓ^∞ -error (Experiment 2)



(c) MMWU weights (Experiment 1)



(f) MMWU weights (Experiment 2)

Figure 1. (a) and (d) plot the spectral gap $1 - |\lambda_2(S)|$ before and after reweighting as a function of n . (b) and (e) plot the relative ℓ^∞ error of the unweighted and weighted spectral methods. (c) and (f) visualize the MMWU weights as heatmaps.